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Some specifics about the general coordinate transformations.

$$\frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x^{\mu'}} = \partial_{\underset{\text{column}}{\mu}} X^{\overset{\text{row}}{\mu'}} \partial_{\underset{\text{column}}{\mu'}} X^{\overset{\text{row}}{\nu}} = \delta_{\mu}{}^{\nu}$$

$$M \quad M^{-1} = \underline{\mathbb{I}}$$

Example: $x^{1'} = x^1 + x^2 \quad x^{2'} = -x^1 + x^2 \Rightarrow \frac{\partial x^{1'}}{\partial x^1} = 1 \quad \frac{\partial x^{1'}}{\partial x^2} = 1 \quad \frac{\partial x^{2'}}{\partial x^1} = -1 \quad \frac{\partial x^{2'}}{\partial x^2} = 1 \Rightarrow \frac{\partial x^{\mu'}}{\partial x^{\mu}} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = M$

$\Downarrow$

$x^1 = \frac{1}{2} x^{1'} - \frac{1}{2} x^{2'} \quad x^2 = \frac{1}{2} x^{1'} + \frac{1}{2} x^{2'} \Rightarrow \frac{\partial x^1}{\partial x^{1'}} = \frac{1}{2} \quad \frac{\partial x^1}{\partial x^{2'}} = -\frac{1}{2} \quad \frac{\partial x^2}{\partial x^{1'}} = \frac{1}{2} \quad \frac{\partial x^2}{\partial x^{2'}} = \frac{1}{2} \Rightarrow M^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}$

$\Downarrow$

By golly, it all checks out!

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So let's compare:

Special Relativity $V^\mu \to V^{\mu'} = \underbrace{\Lambda^{\mu'}{}_{\mu}}_{}\, V^\mu$

constant or "global" transformations on (t,x,y,z)

General Relativity $V^\mu \to V^{\mu'} = \underbrace{\frac{\partial x^{\mu'}}{\partial x^\mu}}\, V^\mu$

this allows "local" or coordinate dependent transformations of (t,x,y,z), i.e. $\frac{\partial x^{\mu'}}{\partial x^\mu}(x^\mu)$

The crucial difficulty we will encounter is that unlike the constant $\Lambda^{\mu'}{}_{\mu}$'s, the $\frac{\partial x^{\mu'}}{\partial x^\mu}(x^\mu)$ <u>cannot</u> in general be moved past derivatives! Why does this matter?

Consider the derivative of a tensor:

$$\partial_\mu T_\nu \to \partial_{\mu'} T_{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \left(\frac{\partial x^\nu}{\partial x^{\nu'}} T_\nu \right) = \underbrace{\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \partial_\mu T_\nu}_{\text{This is all we should get } \underline{\text{if}}\ \partial_\mu T_\nu \text{ was a tensor, but...}} + T_\nu \frac{\partial x^\mu}{\partial x^{\mu'}} \underbrace{\partial_\nu \left(\frac{\partial x^\nu}{\partial x^{\nu'}} \right)}_{\text{This is only zero if } \frac{\partial x^\nu}{\partial x^{\nu'}} \text{ is constant.}}$$

So in contrast to SR, the derivative of a tensor is generally not itself a tensor. But we need derivatives for physics and we need tensor equations $\Rightarrow$ We need a new derivative!

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Metrics, flatness and LICs

Recall that the metric provides a one-to-one correspondence between vectors and dual vectors, i.e. it raises or lowers indices while maintaining the underlying tensor.

$g_{\mu\nu} T^{\nu\beta} = T_\mu{}^\beta$ (in contrast to $H_{\mu\nu} T^{\nu\beta} = J_\mu{}^\beta$)

In particular $g^{\alpha\lambda} g_{\lambda\nu} = \delta^\alpha{}_\nu$ $\underbrace{(\text{or } g^{\alpha\lambda} = (g_{\lambda\nu})^{-1})}_{\text{Contrast w/ } T^\mu \neq (T_\mu)^{-1}}$ sets it apart.

Quite often we will express the metric by a line element:

$\mathbb{R}^3$ w/ (r,θ,ϕ) $\Rightarrow$ $g_{\mu\nu} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{pmatrix} \Rightarrow ds^2 = dx^\mu g_{\mu\nu} dx^\nu$

$$= (dr \; d\theta \; d\phi) \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix}$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta \, d\phi^2$$

Careful, $ds^2 \Rightarrow g_{\mu\nu}$ <u>not</u> $g^{\mu\nu}$ (which in this case is $g^{\mu\nu} = \begin{pmatrix} 1 & & \\ & r^{-2} & \\ & & r^{-2}\sin^{-2}\theta \end{pmatrix}$)

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A single space can admit many different metrics (from different coordinate choices).

We know that flat space, e.g. $\mathbb{R}^3$ can have $g_{\mu\nu} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ w/ (x,y,z),

but we could have a more complicated $g_{\mu\nu}$ for the same space (think (r,θ,ϕ)).

So an important question is "how do we figure out if a space is curved?"
This is hard to answer just by looking at the form of the metric, e.g.

$\mathbb{R}^3$ (r,θ,ϕ) $\Rightarrow g_{\mu\nu} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{pmatrix}$ Flat!

S^2 (θ,ϕ) $\Rightarrow g_{\mu\nu} = \begin{pmatrix} R^2 & \\ & R^2\sin^2\theta \end{pmatrix}$ Curved!

We will eventually get a good measure for curvature, but to get a hint at what it contains we first recall that any manifold can appear flat in the neighborhood (small region) around a point. This means that by choosing the right coordinates the metric can be brought to the form

$g_{\mu\nu} \simeq \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{pmatrix}$ and $\partial_\lambda g_{\mu\nu} \simeq 0$ near the point in question.

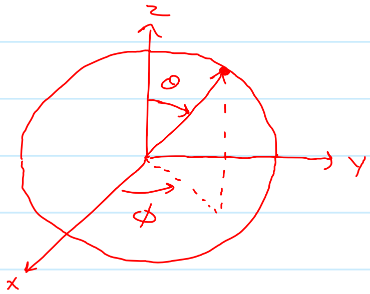
or

$g_{\mu\nu} \simeq \begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots \end{pmatrix}$

The coordinates that do this are called Local Inertial Coordinates.

Example of LIC:

Consider S^2 w/ (θ, ϕ)



$x = R\sin\theta\cos\phi$

$y = R\sin\theta\sin\phi$

$z = R\cos\theta$

$$ds^2_{\theta\phi} = R^2 d\theta^2 + R^2\sin^2\theta\, d\phi^2 \Rightarrow g_{\mu\nu} = \begin{pmatrix} R^2 & \\ & R^2\sin^2\theta \end{pmatrix}$$

Let's focus on the north pole, i.e. $\theta \simeq 0 \Rightarrow g_{\mu\nu} \simeq \begin{pmatrix} R^2 & \\ & 0 \end{pmatrix}$ which is horribly degenerate

Let's find better coordinates for this point: $x = R\theta\cos\phi$, $y = R\theta\sin\phi$ } Near the north pole we won't move in z, and since $\theta \simeq 0$ we use small angle approx. $\sin\theta \simeq \theta$.

In lecture

Inverting: $\theta = \dfrac{\sqrt{x^2+y^2}}{R}$ $\qquad \phi = \tan^{-1}\left(\dfrac{y}{x}\right)$

$$d\theta = \frac{1}{R\sqrt{x^2+y^2}}(x\,dx + y\,dy) \qquad d\phi = \frac{1}{x^2+y^2}(x\,dy - y\,dx)$$

Then:
$$\begin{aligned} ds^2_{xy} = &\left(\frac{x^2}{x^2+y^2} + R^2\sin^2\left(\frac{\sqrt{x^2+y^2}}{R}\right)\frac{y^2}{(x^2+y^2)^2}\right)dx^2 \\ &+ \left(\frac{y^2}{x^2+y^2} + R^2\sin^2\left(\frac{\sqrt{x^2+y^2}}{R}\right)\frac{x^2}{(x^2+y^2)^2}\right)dy^2 \\ &+ 2\left(\frac{xy}{x^2+y^2} - R^2\sin^2\left(\frac{\sqrt{x^2+y^2}}{R}\right)\frac{xy}{(x^2+y^2)^2}\right)dx\,dy \end{aligned}$$

Using: $$\sin\left(\frac{\sqrt{x^2+y^2}}{R}\right) \simeq \frac{\sqrt{x^2+y^2}}{R} - \frac{1}{6}\frac{(x^2+y^2)^{3/2}}{R^3} + \cdots$$

We have: $$ds^2_{xy} = \left(1 - \frac{2y^2}{3R^2} + \cdots\right)dx^2 + \left(1 - \frac{2x^2}{3R^2} + \cdots\right)dy^2 + \left(\frac{4xy}{3R^2} + \cdots\right)dx\,dy$$

Or: $$g_{\mu\nu} = \begin{pmatrix} 1 - \frac{2y^2}{3R^2} & \frac{2xy}{3R^2} \\ \frac{2xy}{3R^2} & 1 - \frac{2x^2}{3R^2} \end{pmatrix} + \text{h.c.}$$

Note: $g_{\mu\nu}(x=y=0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\partial_\lambda g_{\mu\nu}\big|_{x=y=0} = 0$

But: In general the second derivatives $\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial x \partial y}$ will not vanish even when $x=y=0$! These are the quantities from which we will build a good measure of curvature!

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In addition to pointing out what to look for in measuring curvature, LIC's have the following immensely important use:

Ask a question $\longrightarrow$ answer in LIC's (usually easy) $\longrightarrow$ express answer w/ tensors

$\downarrow$

answer good in <u>any</u> coordinates !!

(similar to rest frame use in SR!)

But this is why we have to be careful, e.g. $\partial_\mu T_\nu$ is not a tensor!